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Magnetoelectric Green's functions and their application to the inclusion and inhomogeneity problems

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Abstract

Explicit expressions of magnetoelectric Green's functions are obtained for a transversely isotropic medium exhibiting coupling between the static electric and magnetic fields utilizing the contour integral representation. Four Green's functions exist which represent the coupled static electric and magnetic response to a unit point electric or magnetic charge. The Green's functions are applied to analyze the inclusion and inhomogeneity problems in an infinite magnetoelectric medium, and explicit, closed form expressions are obtained for the Eshelby type tensors. The magnetoelectric Eshelby's tensors can be readily used in the solution of numerous problems in the mechanics and physics of magnetoelectric solids.

Keywords: Magnetoelectric effect; Green's function; Inclusion; Inhomogeneity

1. Introduction

Green's functions are one of the well-established tools in the solution of numerous problems in the mechanics and physics of solids. Evaluation and application of elastic Green's function in isotropic and anisotropic media are included in the review article of Bacon et al. (1978) and the text of Mura (1987). Application of Green's function in condensed matter and solid state physics can be found in Doniach and Sondheimer (1974) and Rickayzen (1980). The key contributions to the study of elastostatic Green's function were made by Freedholm (1900), Lifshitz and Rozentsveig (1947), Kröner (1953), Synge (1957), Willis (1965), Mura and Kinoshita (1971), and Pan and Chou (1976). Recently, the electroelastic Green's functions for a piezoelectric solid were studied by Deeg (1980), Wang (1992), Chen (1993), Dunn (1994), Dunn and Wienecke (1996), Akamatsu and Tanuma (1997), Michelitsch (1997), Gao and Fan (1998), and Karapetian et al. (2000).

Most of the previous studies of Green's functions for a medium with coupled-field behavior focused on the piezoelectric solid where the elastic and electric fields are coupled. Very limited work has been directed

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toward the analysis of Green's functions in the magnetoelectric solid where the static electric and magnetic fields are coupled. Yet the magnetoelectric coupling has both theoretical and practical significance in solid state physics and materials science. Though first predicted by Pierre Curie, magnetoelectric coupling was originally thought to be forbidden because it violates time-reversal symmetry, until Laudau and Lifshitz (1960) pointed out that time reversal is not a symmetry operation in some magnetic crystals. Based on this argument, Dzyaloshinskii (1960) predicted that magnetoelectric effect should occur in antiferromagnetic crystal Cr_2O_3, which was verified experimentally by Astrov (1960, 1961). Since then the magnetoelectric coupling has been observed in single-phase materials where simultaneous electric and magnetic ordering coexists, and in two-phase composites where the participating phases are piezoelectric and piezomagnetic (Van Run et al., 1974; Bracke and Van Vliet, 1981). Agyei and Birman (1990) carried out a detailed analysis of the linear magnetoelectric effect, which showed that the effect should occur not only in some magnetic but also in some electric crystals. Pradhan (1993) showed that an electric charge placed in a magnetoelectric medium becomes a source of induced magnetic field with non-zero divergence of volume integral. Magnetoelectric effect in two-phase composites has been analyzed by Harshe et al. (1993a,b), Avellaneda and Harshe (1994), Nan (1994), Benveniste (1995), Li and Dunn (1998a,b), Li (2000), and Nan et al. (2001). Broadband transducers based on magnetoelectric effect have also been developed (Bracke and Van Vliet, 1981).

The Green's functions for the magnetoelectroelastic solid have been studied by several authors. Chung and Ting (1995) showed that the Green's function for an elliptic hole or rigid inclusion in an anisotropic elastic medium can be modified easily for an anisotropic medium with piezoelectric, piezomagnetic and magnetoelectric coupling by extending the Stroh's formalism to a ten-dimensional formalism. Kirchner and Alshits (1996) obtained the fields in a wedge subjected to various boundary conditions in which the anisotropic elastic, piezoelectric, piezomagnetic, and magnetoelectric constants show an angular variation. Liu et al. (2001) obtained Green's functions for an infinite two-dimensional anisotropic magnetoelectroelastic medium containing an elliptical cavity. In this work, we study the Green's function for a magnetoelectric solid without piezoelectric and piezomagnetic coupling, which simplifies the analysis considerably and allows us to obtain the explicit expressions.

Although virtually all magnetoelectric materials of practical significance are piezoelectric-piezomagnetic composites with inherent electroelastic and magnetoelastic coupling, it is possible to fabricate such composites without macroscopic piezoelectric and piezomagnetic effects by controlling the texture of the composites, because they are forbidden in materials having a symmetry center. This could be accomplished by randomly mixing piezoelectric and piezomagnetic phases at grain level to achieve center symmetry. In fact, suppressing the macroscopic electroelastic and magnetoelastic coupling is desirable from the device point of view, because it suppresses the associated macroscopic stresses and strains, which may complicate magnetoelectric device design and operation. Explicit expressions for magnetoelectric Green's function, which could be used for the analysis of such materials, thus are desirable. In this work we derive the explicit expressions of magnetoelectric Green's functions for a transversely isotropic medium exhibiting full coupling between the static electric and magnetic fields utilizing the contour integral. The Green's functions are then applied to analyze the magnetoelectric inclusion and inhomogeneity problems in an infinite medium, and explicit, closed form expressions are obtained for magnetoelectric Eshelby's tensors. The Eshelby's tensors serve as cornerstone in the micromechanis modeling of heterogeneous materials with coupled-field behaviors, and can be readily used in the solution of numerous problems in the mechanics and physics of heterogeneous solids.

2. Static linear magnetoelectricity

We consider a medium that exhibits linear, static, and anisotropic coupling between the electric and magnetic fields, with the constitutive equations given by:

$$D_i = \kappa_{il}E_l + a_{il}H_l,$$
$$B_i = a_{il}E_l + \mu_{il}H_l, \tag{1a}$$

where D_i and E_i are the electric displacement and field; B_i and H_i are the magnetic flux and field; κ_{il} and μ_{il} are the dielectric and magnetic permittivity; and a_{il} is the magnetoelectric coefficient coupling the static electric and magnetic fields. In the stationary case where there is no free electric charge and current, electric displacement and magnetic flux satisfy the Gauss equations,

$$D_{i,i} = 0,$$
$$B_{i,i} = 0, \tag{2a}$$

where the subscript ",i" is used to denote a partial differentiation with respect to x_i. The electric and magnetic fields, on the other hand, can be derived from scalar electric and magnetic potentials,

$$E_i = -\phi_{,i},$$
$$H_i = -\varphi_{,i}. \tag{3a}$$

While the electric potential ϕ has clear physical interpretation, the magnetic potential φ is introduced for mathematical convenience.

In order to treat the electric and magnetic variables on equal footing, we introduce a short notation, which is analog of that introduced by Barnett and Lothe (1975) for piezoelectricity. This notation is identical to conventional indicial notation with the exception that both lower case and upper case subscripts are used. Lowercase subscripts take on the range $1 \to 3$ as usual, while uppercase subscripts take on the range $1 \to 2$, and repeated uppercase subscripts are summed over $1 \to 2$. With this notation, the magnetoelectric field variables take the following forms:

$$\Sigma_{iJ} = \begin{cases} D_i & J=1, \\ B_i & J=2, \end{cases} \quad Z_{Ji} = \begin{cases} E_i & J=1, \\ H_i & J=2, \end{cases} \quad U_J = \begin{cases} -\phi & J=1, \\ -\varphi & J=2, \end{cases} \tag{4a}$$

and the magnetoelectric moduli are expressed as:

$$\widehat{E}_{iJMn} = \begin{cases} \kappa_{in} & J=1, \quad M=1, \\ a_{in} & J=1, \quad M=2, \\ a_{in} & J=2, \quad M=1, \\ \mu_{in} & J=2, \quad M=2. \end{cases} \tag{4b}$$

It is noted that the uppercase subscript 1 and 2 is reserved for electric and magnetic variables, respectively. Under this notation, the constitutive equations, Gaussian equations, and gradient equations can be written as:

$$\Sigma_{iJ} = \widehat{E}_{iJMn}Z_{Mn}, \tag{1b}$$

$$\Sigma_{iJ,i} = 0, \tag{2b}$$

$$Z_{Ji} = U_{J,i}. \tag{3b}$$

It is worthwhile to notice that Σ_{iJ}, Z_{Mn}, U_J, and $\widehat{E}_{iJMn}$ are not tensors, and transformation between different coordinate systems must be performed on individual tensors according to corresponding tensor transformation laws.

3. Magnetoelectric Green's functions

3.1. Contour integral representation

The magnetoelectric Green's functions $G_{MR}(\mathbf{x}-\mathbf{x}')$ are defined through the following partial differential equation

$$\widehat{E}_{iJMn}G_{MR,ni}(\mathbf{x}-\mathbf{x}')+\delta_{JR}\delta(\mathbf{x}-\mathbf{x}')=0, \tag{5}$$

where $\delta(\mathbf{x}-\mathbf{x}')$ is the three-dimensional Dirac delta function, and δ_{JR} is the generalized Kronecker delta. They have the following physical interpretation:

$G_{11}(\mathbf{x}-\mathbf{x}')$: the electric potential at $\mathbf{x}$ due to a unit point electric charge at $\mathbf{x}'$;
$G_{12}(\mathbf{x}-\mathbf{x}')$: the electric potential at $\mathbf{x}$ due to a unit point magnetic charge at $\mathbf{x}'$;
$G_{21}(\mathbf{x}-\mathbf{x}')$: the magnetic potential at $\mathbf{x}$ due to a unit point electric charge at $\mathbf{x}'$;
$G_{22}(\mathbf{x}-\mathbf{x}')$: the magnetic potential at $\mathbf{x}$ due to a unit point magnetic charge at $\mathbf{x}'$.

Note that the magnetic charge, or magnetic monopole, is introduced for mathematical convenience, which simplifies the analysis of magnetoelectric inclusion and inhomogeneity problems, as we show later on.

In order to determine the magnetoelectric Green's functions $G_{MR}(\mathbf{x}-\mathbf{x}')$, we take the Radon integral transform (Gel'Fand et al., 1966), $\tilde{f}(\mathbf{z},\alpha)=\int\int_{\mathbf{z}\cdot\mathbf{x}=\alpha}f(\mathbf{x})\,\mathrm{d}S(\mathbf{x})$, on Eq. (5) (also see Bacon et al., 1978; Deeg, 1980; Dunn, 1994),

$$\int\int_{\mathbf{z}\cdot\mathbf{x}=\alpha}[\widehat{E}_{ijMn}G_{MR,ni}(\mathbf{x}-\mathbf{x}')+\delta_{JR}\delta(\mathbf{x}-\mathbf{x}')]\,\mathrm{d}S(\mathbf{x})=0, \tag{6}$$

where the transform space variables $\mathbf{z}$ and α are vector and scalar, respectively, and the integration is performed over the infinite plane $\mathbf{z}\cdot\mathbf{x}=\alpha$. Denoting the Radon transform of Green's function by $\widetilde{G}_{MR}$, and using the following properties of the Radon transform,

$$\int\int_{\mathbf{z}\cdot\mathbf{x}=\alpha}f(\mathbf{x}-\mathbf{x}')\,\mathrm{d}S(\mathbf{x})=\tilde{f}(\mathbf{z},\alpha-\mathbf{z}\cdot\mathbf{x}'),$$

$$\int\int_{\mathbf{z}\cdot\mathbf{x}=\alpha}f_{,i}(\mathbf{x})\,\mathrm{d}S(\mathbf{x})=z_i\frac{\partial\tilde{f}(\mathbf{z},\alpha)}{\partial\alpha},$$

$$\int\int_{\mathbf{z}\cdot\mathbf{x}=\alpha}\delta(\mathbf{x})\,\mathrm{d}S(\mathbf{x})=\delta(\alpha),$$

we can rewrite Eq. (6) as

$$\widehat{E}_{iJMn}z_iz_n\frac{\partial^2\widetilde{G}_{MR}(\mathbf{z},\alpha-\mathbf{z}\cdot\mathbf{x}')}{\partial\alpha^2}+\delta_{JR}\delta(\alpha-\mathbf{z}\cdot\mathbf{x}')=0. \tag{7}$$

Multiplying the inverse of $K_{JM}(\mathbf{z})=\widehat{E}_{iJMn}z_iz_n$ at both sides of Eq. (7), we obtain

$$\frac{\partial^2\widetilde{G}_{MR}(\mathbf{z},\alpha-\mathbf{z}\cdot\mathbf{x}')}{\partial\alpha^2}=-K^{-1}_{MR}(\mathbf{z})\delta(\alpha-\mathbf{z}\cdot\mathbf{x}'). \tag{8}$$

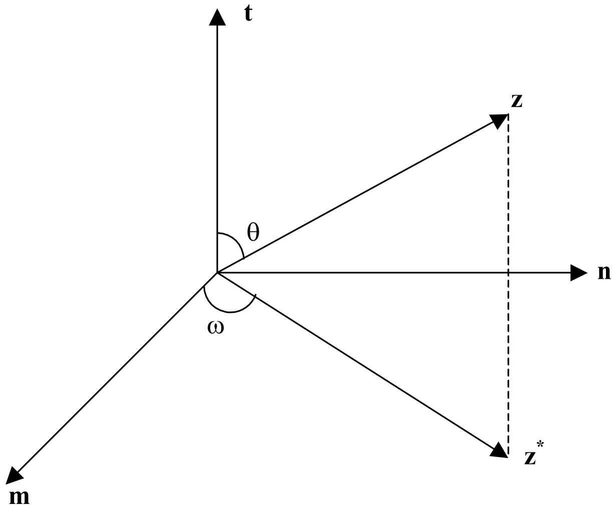

Fig. 1. Relationship between vectors **t**, **z**, and **z***.

Taking the inverse Radon transform

$$f(\mathbf{x}) = -\frac{1}{8\pi^2} \oiint_{|\mathbf{z}|=1} \left[\frac{\partial^2 \tilde{f}(\mathbf{z},\alpha)}{\partial \alpha^2} \right]_{\mathbf{z}\cdot\mathbf{x}=\alpha} \mathrm{d}S(\mathbf{z}) \tag{9}$$

on Eq. (8), we obtain

$$G_{MR}(\mathbf{x}-\mathbf{x}') = \frac{1}{8\pi^2} \oiint_{|\mathbf{z}|=1} K_{MR}^{-1}(\mathbf{z})\delta[\mathbf{z}\cdot(\mathbf{x}-\mathbf{x}')]\,\mathrm{d}S(\mathbf{z}). \tag{10}$$

Setting $\mathbf{x}-\mathbf{x}' = |\mathbf{x}-\mathbf{x}'|\mathbf{t}$, where $\mathbf{t}$ is the unit vector along $\mathbf{x}-\mathbf{x}'$-direction, and utilizing the property of Dirac delta function, we obtain

$$G_{MR}(\mathbf{x}-\mathbf{x}') = \frac{1}{8\pi^2|\mathbf{x}-\mathbf{x}'|} \oiint_{|\mathbf{z}|=1} [K_{MR}^{-1}(\mathbf{z})\delta(\mathbf{z}\cdot\mathbf{t})]\,\mathrm{d}S(\mathbf{z}). \tag{11}$$

The relationship between vectors $\mathbf{z}$ and $\mathbf{t}$ is shown in Fig. 1, with $\mathbf{z}^*$ lies in the plane $\mathbf{m}$–$\mathbf{n}$ normal to $\mathbf{t}$, i.e., $\mathbf{z}^*\cdot\mathbf{t}=0$, and $\theta(\mathbf{z}^*)=\pi/2$. Utilizing the property of Dirac delta function again, we can rewrite Eq. (11) as

$$G_{MR}(\mathbf{x}-\mathbf{x}') = \frac{1}{8\pi^2|\mathbf{x}-\mathbf{x}'|} \oint_{|\mathbf{z}=1|} K_{MR}^{-1}(\mathbf{z}^*)\,\mathrm{d}\omega(\mathbf{z}^*). \tag{12}$$

Note that $|\mathbf{z}|=1$ in the integral represents a unit circle, which is the intersection of the unit sphere with plane $\mathbf{m}$–$\mathbf{n}$, as shown in Fig. 1. It is analogous to the contour integral obtained by Synge (1957) for anisotropy elasticity, and that obtained by Deeg (1980) for piezoelectricity. In general, it needs to be evaluated numerically.

3.2. Explicit expressions for transversely isotropic media

Now we consider a transversely isotropic medium, characterized by dielectric constants $\kappa_{11}=\kappa_{22}$ and κ_{33}, magnetic constants $\mu_{11}=\mu_{22}$ and μ_{33}, and magnetoelectric constants $a_{11}=a_{22}$ and a_{33}; all other material constants are zero. The 2×2 matrix K_{JM} for the transversely isotropic medium can be expressed as

$$K_{JM} = \begin{bmatrix} \kappa_{11}(z_1^2 + z_2^2) + \kappa_{33} z_3^2 & a_{11}(z_1^2 + z_2^2) + a_{33} z_3^2 \\ a_{11}(z_1^2 + z_2^2) + a_{33} z_3^2 & \mu_{11}(z_1^2 + z_2^2) + \mu_{33} z_3^2 \end{bmatrix} \tag{13}$$

and its inverse can be represented by

$$K_{JM}^{-1} = \frac{L_{JM}}{g} = \frac{1}{g}\begin{bmatrix} \mu_{11}(z_1^2 + z_2^2) + \mu_{33} z_3^2 & -a_{11}(z_1^2 + z_2^2) - a_{33} z_3^2 \\ -a_{11}(z_1^2 + z_2^2) - a_{33} z_3^2 & \kappa_{11}(z_1^2 + z_2^2) + \kappa_{33} z_3^2 \end{bmatrix}, \tag{14}$$

with

$$g = (\kappa_{33}\mu_{33} - a_{33}^2)[A_1(z_1^2 + z_2^2) + z_3^2][A_2(z_1^2 + z_2^2) + z_3^2] \tag{15}$$

and

$$A_1 = \frac{2a_{11}a_{33} - \kappa_{33}\mu_{11} - \kappa_{11}\mu_{33} - \sqrt{(\kappa_{33}\mu_{11} + \kappa_{11}\mu_{33} - 2a_{11}a_{33})^2 - 4(a_{11}^2 - \kappa_{11}\mu_{11})(a_{33}^2 - \kappa_{33}\mu_{33})}}{2(a_{33}^2 - \kappa_{33}\mu_{33})}, \tag{16a}$$

$$A_2 = \frac{2a_{11}a_{33} - \kappa_{33}\mu_{11} - \kappa_{11}\mu_{33} + \sqrt{(\kappa_{33}\mu_{11} + \kappa_{11}\mu_{33} - 2a_{11}a_{33})^2 - 4(a_{11}^2 - \kappa_{11}\mu_{11})(a_{33}^2 - \kappa_{33}\mu_{33})}}{2(a_{33}^2 - \kappa_{33}\mu_{33})}. \tag{16b}$$

To simplify the evaluation of Green's functions, we assume the source point is located at the origin, $\mathbf{x}' = (0,0,0)$, and the observing point is lying in the x_1–x_3 plane, $\mathbf{x} = (x_1, 0, x_3)$, with $\tan\theta = x_1/x_3$. Expressions for any other source and observation points can be easily obtained from the transversely isotropic symmetry of the medium. For such configuration, the intersection between the unit sphere $|\mathbf{z}| = 1$ and the plane **m**–**n** is a unit circle represented by $\eta_1^2 + \eta_2^2 = 1$, with $\eta_3 = 0$ corresponding to the plane **m**–**n**; see Fig. 1. With no loss of generality, η_2 is chosen so as to coincide with a unit vector in the x_2-direction. Thus we have

$$z_1 = \eta_1 \cos\theta, \quad z_2 = \eta_2, \quad z_3 = -\eta_1 \sin\theta. \tag{17}$$

After introducing a complex variable $\zeta = \eta_1 + \mathrm{i}\eta_2 = \mathrm{e}^{\mathrm{i}\omega}$, we obtain

$$\eta_1 = \frac{\zeta + 1/\zeta}{2}, \quad \eta_2 = \frac{\zeta - 1/\zeta}{2\mathrm{i}}, \quad \mathrm{d}\omega = \frac{\mathrm{d}\zeta}{\mathrm{i}\zeta}. \tag{18}$$

We can then express the integrals in Eq. (12) in terms of complex variable ζ, and evaluate them using Cauchy's residue theorem. With those transformation and when $\theta \neq 0$, g can be written as

$$g = \frac{(\kappa_{33}\mu_{33} - a_{33}^2)\sin^4\theta}{16\zeta^4}(1 - A_1)h_1(\zeta)(1 - A_2)h_2(\zeta), \tag{19}$$

with

$$h_i(\zeta) = \zeta^4 + 2B_i\zeta^2 + 1, \qquad B_i = \frac{A_i\cos^2\theta + \sin^2\theta + A_i}{(1 - A_i)\sin^2\theta}. \tag{20}$$

It is noted that if ζ is a root of $h_i(\zeta) = 0$, so are $-\zeta$, $1/\zeta$, and $-1/\zeta$, so in general, there are two roots, $\pm\chi_i$, with moduli less than unity, and two roots, $\pm\beta_i$, with moduli greater than unity. Eq. (19) can then be rewritten as

$$g = \frac{(\kappa_{33}\mu_{33} - a_{33}^2)\sin^4\theta}{16\zeta^4}(1 - A_1)(1 - A_2)(\zeta^2 - \chi_1^2)(\zeta^2 - \beta_1^2)(\zeta^2 - \chi_2^2)(\zeta^2 - \beta_2^2). \tag{21}$$

Substituting Eq. (21) into Eq. (12), we obtain

$$G_{MR}(\mathbf{x} - \mathbf{x}') = C\oint_{|\mathbf{z}=1|} \frac{\zeta^3 L_{MR}(\zeta)}{(\zeta^2 - \chi_1^2)(\zeta^2 - \beta_1^2)(\zeta^2 - \chi_2^2)(\zeta^2 - \beta_2^2)}\mathrm{d}\zeta, \tag{22}$$

with

$$L_{MR}(\zeta) = \frac{1}{4\zeta^2}\begin{cases} (1+\zeta^2)^2(\mu_{11}\cos^2\theta + \mu_{33}\sin^2\theta) - (1-\zeta^2)^2\mu_{11}, & MR = 11 \\ -(1+\zeta^2)^2(a_{11}\cos^2\theta + a_{33}\sin^2\theta) + (1-\zeta^2)^2 a_{11}, & MR = 12 \\ -(1+\zeta^2)^2(a_{11}\cos^2\theta + a_{33}\sin^2\theta) + (1-\zeta^2)^2 a_{11}, & MR = 21 \\ (1+\zeta^2)^2(\kappa_{11}\cos^2\theta + \kappa_{33}\sin^2\theta) - (1-\zeta^2)^2\kappa_{11}, & MR = 22 \end{cases} \tag{23a}$$

and

$$C = \frac{2}{\mathrm{i}\pi^2|\mathbf{x}|(\kappa_{33}\mu_{33} - a_{33}^2)\sin^4\theta(1 - A_1)(1 - A_2)}. \tag{23b}$$

Notice that the limiting case $|\chi_i| = |\beta_i| = 1$ cannot occur because $g \neq 0$ for $\mathbf{z} \neq 0$ (Willis, 1965), and the integrands are never singular on the contour of integration $|\mathbf{z}| = 1$. Now define

$$t_{MR}(\zeta) = \frac{\zeta^3 L_{MR}(\zeta)}{(\zeta^2 - \beta_1^2)(\zeta^2 - \beta_2^2)}, \tag{24}$$

which is analytic inside $|\zeta| = 1$ since $|\beta_i| > 1$, Eq. (22) can be written as

$$G_{MR}(\mathbf{x} - \mathbf{x}') = C\oint_{|\mathbf{z}=1|} \frac{t_{MR}(\zeta)}{(\zeta^2 - \chi_1^2)(\zeta^2 - \chi_2^2)}\mathrm{d}\zeta \tag{25}$$

and finally, according to Cauchy's residue theorem, the Green's function can be evaluated as

$$G_{MR}(\mathbf{x} - \mathbf{x}') = \frac{4}{\pi|\mathbf{x}|(\kappa_{33}\mu_{33} - a_{33}^2)\sin^4\theta(1 - A_1)(1 - A_2)}\left[\frac{t_{MR}(\chi_1) - t_{MR}(\chi_2)}{(\chi_1^2 - \chi_2^2)}\right]. \tag{26}$$

Expression for $\theta = 0$ can be obtained as limiting form of the above expressions (Willis, 1965). Finally, we note that for isotropic media, in which $A_1 = A_2 = 1$, and K_{MR}^{-1} does not depend on $\mathbf{z}$, we can evaluate the contour integral in Eq. (12) directly.

4. Magnetoelectric inclusion and inhomogeneity problems

4.1. Magnetoelectric inclusion problem

The magnetoelectric Green's functions can be used to solve the inclusion and inhomogeneity problems in a magnetoelectric solid. Adopting Mura's (1987) terminology, we denote an inclusion as a subdomain Ω in an infinite matrix D with the same magnetoelectric moduli $\widehat{E}_{iJKl}$ as that of the matrix, but undergoing an eigenfield Z_{Kl}^{T}. An inhomogeneity is a subdomain Ω in an infinite matrix D with a different magnetoelectric moduli, $\widehat{E}'_{iJKl}$, from that of the matrix, $\widehat{E}_{iJKl}$. The eigenfield Z_{Kl}^{T} in the inclusion, for example, can be caused by the spontaneous electric polarization and magnetic moment, which occur during a crystallographic phase transformation. It is that which would occur if Ω were unconstrained by D. Actual constrained magnetoelectric field inside the inclusion is in general a function of material moduli of the matrix, the shape and orientation of the inclusion, and the distribution of eigenfield in the inclusion. The disturbance field caused by an inclusion or inhomogeneity is known as the depolarization and demagnetizing field.

With the presence of an eigenfield Z_{Kl}^{T}, the constitutive equation and equilibrium equation for the inclusion need to be rewritten as

$$\Sigma_{iJ} = \widehat{E}_{iJMn}(Z_{Mn} - Z_{Mn}^{\mathrm{T}}) \tag{27}$$

and

$$\widehat{E}_{iJMn}Z_{Mn,i} = \widehat{E}_{iJMn}Z_{Mn,i}^{\mathrm{T}}. \tag{28}$$

From Eq. (28), it is found that $\widehat{E}_{iJMn}Z_{Mn,i}^{\mathrm{T}}$ functions as electric charge and magnetic monopole. While $\widehat{E}_{iJMn}Z_{Mn,i}^{\mathrm{T}}$ is finite within inclusion Ω, it behaves as a delta function across inclusion surface $\partial\Omega$, which is equivalent to a thin layer of concentrated charges around $\partial\Omega$, representing a jump in $\Sigma_{iJ}^{\mathrm{T}} = \widehat{E}_{iJMn}Z_{Mn}^{\mathrm{T}}$ across this boundary, $[\Sigma_{iJ}^{\mathrm{T}}]$. The overall effect, therefore, is represented by the additional flux acting on Ω over its boundary $\partial\Omega$, $q_J = n_i[\Sigma_{iJ}^{\mathrm{T}}] = -n_i\Sigma_{iJ}^{\mathrm{T}}$, where n_i is the unit surface normal pointing outward. The resulting electric and magnetic potentials in the infinite body D are then produced by $\widehat{E}_{iJMn}Z_{Mn,i}^{\mathrm{T}}$ distributed within inclusion Ω, and the flux $q_J(\mathbf{x})$ acting upon inclusion surface $\partial\Omega$,

$$U_M(\mathbf{x}) = \int\int_{\partial\Omega} G_{MJ}(\mathbf{x}-\mathbf{x}')\Sigma_{iJ}^{\mathrm{T}}(\mathbf{x}')n_i(\mathbf{x}')\,\mathrm{d}S(\mathbf{x}') - \int\int_{\Omega}\int G_{MJ}(\mathbf{x}'-\mathbf{x}')\Sigma_{iJ,i'}^{\mathrm{T}}(\mathbf{x}')\,\mathrm{d}V(\mathbf{x}'), \tag{29a}$$

which can be simplified by Gauss theorem,

$$U_M(\mathbf{x}) = -\int\int_{\Omega}\int G_{MJ,i}(\mathbf{x}-\mathbf{x}')\Sigma_{iJ}^{T}(\mathbf{x}')\,\mathrm{d}V(\mathbf{x}'). \tag{29b}$$

The magnetoelectric field can then determined to be

$$U_{M,n}(\mathbf{x}) = -\int\int_{\Omega}\int G_{MJ,in}(\mathbf{x}-\mathbf{x}')\Sigma_{iJ}^{\mathrm{T}}(\mathbf{x}')\,\mathrm{d}V(\mathbf{x}'). \tag{30}$$

From the analysis we find that although the magnetic charge, or magnetic monopole, does not exist, it can be conveniently used to represent the discontinuity of magnetic moment at the inclusion boundary, thus help to solve the inclusion problem in a magnetoelectric medium.

In order to determine the magnetoelectric field due to the eigenfield in a inclusion from Eqs. (29a), (29b) and (30), we need the derivatives of Green's functions. To this end, we differentiate Eq. (10) to obtain

$$G_{MR,i}(\mathbf{x}-\mathbf{x}') = \frac{1}{8\pi^2}\oint\!\!\oint_{|\mathbf{z}=1|} z_i K_{MR}^{-1}(\mathbf{z})\delta'[\mathbf{z}\cdot(\mathbf{x}-\mathbf{x}')]\,\mathrm{d}S(\mathbf{z}) \tag{31}$$

and

$$G_{MR,in}(\mathbf{x}-\mathbf{x}') = \frac{1}{8\pi^2}\oint\!\!\oint_{|\mathbf{z}=1|} z_i z_n K_{MR}^{-1}(\mathbf{z})\delta''[\mathbf{z}\cdot(\mathbf{x}-\mathbf{x}')]\,\mathrm{d}S(\mathbf{z}). \tag{32a}$$

Defining $\nabla_x^2 = \partial^2/\partial x_s\partial x_s$, and taking into account the fact that $z_s z_s = 1$, we can rewrite Eq. (32a) as

$$G_{MJ,in}(\mathbf{x}-\mathbf{x}') = \frac{1}{8\pi^2}\nabla_x^2\oint\!\!\oint_{|\mathbf{z}|=1} z_i z_n K_{MJ}^{-1}\delta[\mathbf{z}\cdot(\mathbf{x}-\mathbf{x}')]\,\mathrm{d}S(\mathbf{z}), \tag{32b}$$

with which Eq. (30) can be rewritten as

$$U_{M,n}(\mathbf{x}) = -\frac{1}{8\pi^2} \oiint_{|\mathbf{z}|=1} z_i z_n K_{MJ}^{-1} \nabla_x^2 \int\int_{\Omega}\int \Sigma_{iJ}^{\mathrm{T}}(\mathbf{x}')\delta[\mathbf{z}\cdot(\mathbf{x}-\mathbf{x}')]\,\mathrm{d}V(\mathbf{x}')\,\mathrm{d}S(\mathbf{z}). \tag{33}$$

Eq. (33) is valid for any arbitrary material symmetry, inclusion shape, and eigenfield distribution.

4.2. Magnetoelectric Eshelby's tensors

To carry on the analysis, let us consider a uniform eigenfield in an ellipsoidal inclusion, specified by

$$\left(\frac{x_1'}{a_1}\right)^2 + \left(\frac{x_2'}{a_2}\right)^2 + \left(\frac{x_3'}{a_3}\right)^2 \leqslant 1, \tag{34a}$$

where a_1, a_2, and a_3 are the dimensions of the inclusion in the x_1'-, x_2'-, and x_3'-directions, respectively. Making the following scaling, $\tau_i' = x_i'/a_i$ (no summation on i), we can rewrite the ellipsoidal equation (34a) as

$$\tau_1'^2 + \tau_2'^2 + \tau_3'^2 \leqslant 1, \tag{34b}$$

with which we obtain

$$\int\int_{\Omega}\int \Sigma_{iJ}^{\mathrm{T}}(\mathbf{x}')\delta[\mathbf{z}\cdot(\mathbf{x}-\mathbf{x}')]\,\mathrm{d}V(\mathbf{x}') = \Sigma_{iJ}^{\mathrm{T}}\frac{a_1a_2a_3}{\mu}\int\int_{\boldsymbol{\tau}'\leqslant 1}\int \delta[\mathbf{s}\cdot(\boldsymbol{\tau}-\boldsymbol{\tau}')]\,\mathrm{d}\tau_1'\tau_2'\tau_3', \tag{35}$$

where $\mathbf{s} = \mathbf{k}/\mu$ is a unit vector in the direction of $k_i = a_i z_i$ (no summation on i). To evaluate the integral in Eq. (35), let us consider a point $\mathbf{x}$ inside the inclusion so that $|\boldsymbol{\tau}| \leqslant 1$, and write $\boldsymbol{\tau}' = T\mathbf{u} + Q\mathbf{s}$, with $\mathbf{u}\cdot\mathbf{s} = 0$, and $Q = \mathbf{s}\cdot\boldsymbol{\tau}$; see Fig. 2. Eq. (35) is then reduced to

$$\int\int_{\Omega}\int \Sigma_{iJ}^{\mathrm{T}}(\mathbf{x}')\delta[\mathbf{z}\cdot(\mathbf{x}-\mathbf{x}')]\,\mathrm{d}V(\mathbf{x}') = \Sigma_{iJ}^{\mathrm{T}}\frac{a_1a_2a_3}{\mu}\int\int_{|\boldsymbol{\tau}'|\leqslant 1}\int \delta[-Ts_iu_i]\,\mathrm{d}\tau_1'\,\tau_2'\tau_3'. \tag{36}$$

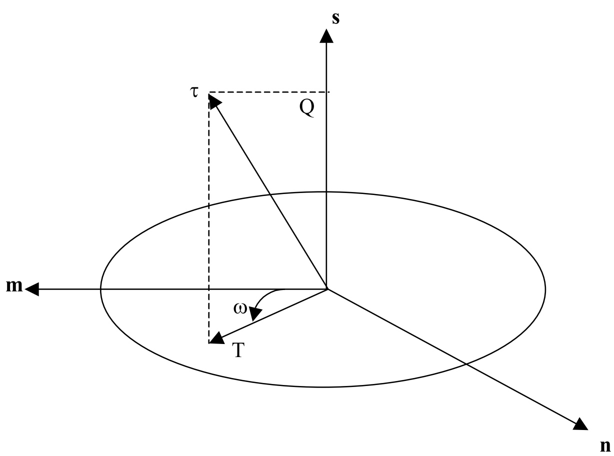

Fig. 2. Relationship between vectors $\mathbf{s}$, $\boldsymbol{\tau}$, and $\mathbf{T}$.

Thus the volume integral over $|\boldsymbol{\tau}'| \leqslant 1$ is reduced to the integral over a plane circular region of radius $\sqrt{1-Q^2} = \sqrt{1-(\mathbf{s}\cdot\boldsymbol{\tau})^2}$ perpendicular to $\mathbf{s}$,

$$\iiint_{\boldsymbol{\tau}' \leqslant 1} \delta[-Ts_i u_i]\,\mathrm{d}\tau_1'\tau_2'\tau_3' = \int_0^{2\pi}\int_0^{\sqrt{1-Q^2}} T\,\mathrm{d}T\,\mathrm{d}\varpi = \pi(1-Q^2) = \pi\left[1-\left(\frac{\mathbf{z}\cdot\mathbf{x}}{\mu}\right)^2\right]. \tag{37}$$

With Eq. (37), the magnetoelectric field inside the inclusion is reduced to

$$U_{M,n}(\mathbf{x}) = \frac{a_1a_2a_3}{4\pi\mu^3}\Sigma_{iJ}^{\mathrm{T}} \oiint_{|\mathbf{z}|=1} z_i z_n K_{MJ}^{-1}\,\mathrm{d}S(\mathbf{z}), \tag{38}$$

which does not depend on the position $\mathbf{x}$, and thus, is uniform inside the inclusion, valid for ellipsoidal inclusions embedded in a magnetoelectric medium with any anisotropy. The Eshelby's tensor can then be defined as

$$Z_{Mn} = S_{MnAb} Z_{Ab}^{\mathrm{T}}, \tag{39}$$

with

$$S_{MnAb} = -\iiint_{\Omega} G_{MJ,in}(\mathbf{x}-\mathbf{x}')\widehat{E}_{iJAb}\,\mathrm{d}V(\mathbf{x}) = \frac{a_1a_2a_3}{4\pi\mu^3}\widehat{E}_{iJAb} \oiint_{|\mathbf{z}|=1} z_i z_n K_{MJ}^{-1}\,\mathrm{d}S(\mathbf{z}). \tag{40}$$

By the following variable transformation, $\mathrm{d}\xi_i = a_i\,\mathrm{d}z_i/\mu$ (no summation on i), we obtain

$$S_{MnAb} = \frac{1}{4\pi}\widehat{E}_{iJAb}\int_{-1}^{1}\int_0^{2\pi} z_i z_n K_{MJ}^{-1}\,\mathrm{d}\theta\,\mathrm{d}\xi_3, \tag{41}$$

with $\xi_1 = \sqrt{1-\xi_3^2}\cos\theta$ and $\xi_2 = \sqrt{1-\xi_3^2}\sin\theta$. Since $z_i z_n K_{mJ}^{-1}$ is a homogeneous polynomial, we can use $z_i = \xi_i/a_i$ in Eq. (41). Eq. (41) is valid for any material symmetry, and need to be evaluated numerically in general. A numerical algorithm is given in Li (2000). For spheroidal inclusions embedded in an isotropic medium, or cylindrical inclusions and penny-shape inclusions embedded in a transversely isotropic medium, we have obtained the following closed form expressions of Eshelby's tensor:

I Oblate spheroid in an isotropic medium: $\alpha a_3 = a_1 = a_2$, $\alpha > 1$

$$S_{1111} = S_{1212} = S_{2121} = S_{2222} = \frac{1}{2}\left[\frac{1}{1-\alpha^2} + \frac{\alpha^2\tan^{-1}(\alpha^2-1)^{1/2}}{(\alpha^2-1)^{3/2}}\right],$$

$$S_{1313} = S_{2323} = \frac{\alpha^2}{\alpha^2-1} - \frac{\alpha^2\tan^{-1}(\alpha^2-1)^{1/2}}{(\alpha^2-1)^{3/2}}.$$

II Sphere in an isotropic medium: $a_3 = \alpha a_1 = \alpha a_2$, $\alpha = 1$

$$S_{1111} = S_{1212} = S_{1313} = S_{2121} = S_{2222} = S_{2323} = 1/3.$$

III Prolate spheroid in an isotropic medium: $a_3 = \alpha a_1 = \alpha a_2$, $\alpha > 1$

$$S_{1111} = S_{1212} = S_{2121} = S_{2222} = \frac{\alpha[\alpha(\alpha^2-1)^{1/2} - \tanh^{-1}(1-1/\alpha^2)^{1/2}]}{2(\alpha^2-1)^{3/2}},$$

$$S_{1313} = S_{2323} = \frac{1}{1-\alpha^2} + \frac{\alpha\tanh^{-1}(1-1/\alpha^2)^{1/2}}{(\alpha^2-1)^{3/2}}.$$

IV Cylindrical inclusion in a transversely isotropic medium: $a_3 \to \infty$, $a_2 = \alpha a_1$

$$S_{1111} = S_{2121} = \frac{\alpha(\alpha - 1)}{\alpha^2 - 1},$$

$$S_{1212} = S_{2222} = \frac{\alpha - 1}{\alpha^2 - 1}.$$

V Penny-shape inclusion in a transversely isotropic medium: $a_3 \to 0$, $a_1 = a_2$

$$S_{1313} = S_{2323} = 1.$$

All other components are zero. It is noted that for the material symmetries and inclusion shapes considered, the Eshelby's tensor is only a function of inclusion shape aspect ratio, and all the coupling terms between electric and magnetic fields are zero. In other word, it does not depend on the material properties of the matrix. Eshelby's tensor is a very important and well-known concept in micromechanics, and its extensive applications can be found in Mura (1987) and Nemat-Nasser and Hori (1993). In magnetic context, it is related to the demagnetizing factor **N**.

4.3. Inhomogeneity problem

Once the solution for the ellipsoidal inclusion is obtained, the solution for the ellipsoidal inhomogeneity easily follows. As shown by Eshelby (1957) in the elastic case, the inhomogeneity can be simulated by an equivalent inclusion. To be specific, consider the infinite solid with moduli $\widehat{E}_{iJKl}$ that contains an ellipsoidal inhomogeneity with moduli $\widehat{E}'_{iJKl}$. In the absence of an applied load, the fields in both the inhomogeneity and the matrix are zero. When subjected to a far-field uniform load Σ^0_{iJ}, the fields $\Sigma^0_{iJ} + \Sigma^{\rm d}_{iJ}$ in the inhomogeneity can be written as

$$\Sigma^0_{iJ} + \Sigma^{\rm d}_{iJ} = \widehat{E}'_{iJMN}(Z^0_{Mn} + Z^{\rm d}_{Mn}) = \widehat{E}_{iJMn}(Z^0_{Mn} + Z^{\rm d}_{Mn} - Z^*_{Mn}). \tag{42}$$

In Eq. (42), Z^0_{Mn} is the uniform field that would exist in the absence of the inhomogeneity, and $Z^{\rm d}_{Mn}$ is the disturbance of the uniform field due to the presence of the inhomogeneity, or the so-called depolarization and demagnetizing field. The first right-hand side of the equation represents the fields in the actual inhomogeneity; the second one represents the fields in an inclusion of the same shape and orientation as the inhomogeneity, but with an eigenfield Z^*_{Mn}, i.e., an equivalent inclusion. Simulation of the inhomogeneity by an equivalent inclusion is possible when an appropriate Z^*_{Mn} can be found to enforce the second equality of the equation, which gives

$$Z^*_{Mn} = -H^{-1}_{MniJ}(\widehat{E}'_{iJAb} - \widehat{E}_{iJAb})Z^0_{Ab}, \tag{43}$$

with

$$H_{iJMn} = (\widehat{E}'_{iJAb} - \widehat{E}_{iJAb})S_{AbMn} + \widehat{E}_{iJMn}. \tag{44}$$

From Eqs. (43), (44), and (39), the concentration factor A_{CdAb} defined by $Z^0_{Cd} + Z^{\rm d}_{Cd} = A_{CdAb}Z^0_{Ab}$ for a single inhomogeneity embedded in infinite matrix easily follows as

$$A_{CdAb} = -S_{CdMn}H^{-1}_{MniJ}(\widehat{E}'_{iJAb} - \widehat{E}_{iJAb}) + I_{CdAb} = [I_{CdAb} + S_{CdMn}\widehat{E}^{-1}_{MniJ}(\widehat{E}'_{iJAb} - \widehat{E}_{iJAb})]^{-1}. \tag{45}$$

The concentration factor is a key concept in the micromechanics modeling of the heterogeneous materials, and is used extensively to predict the effective behavior and analyze the internal field distribution of heterogeneous materials. In the case where there is a prescribed eigenfield $Z^{\rm T}_{Kl}$ in the inhomogeneity, i.e., an inhomogeneous inclusion, the fields are

$$\Sigma^0_{iJ} + \Sigma^{\rm d}_{iJ} = \widehat{E}'_{iJMN}(Z^0_{Mn} + Z^{\rm d}_{Mn} - Z^T_{Mn}) = \widehat{E}_{iJMn}(Z^0_{Mn} + Z^{\rm d}_{Mn} + Z^T_{Mn} - Z^{**}_{Mn}) = \widehat{E}_{iJMn}(Z^0_{Mn} + Z^{\rm d}_{Mn} - Z^*_{Mn}), \tag{46}$$

with which the equivalent eigenfield Z^*_{Mn} can be solved to provide the fields inside the inhomogeneous inclusion.

The above results for the interior fields can be used to obtain the fields just outside an inclusion or inhomogeneity by making use of the continuity conditions on Z_{Mn} and the jump conditions on U_M at the inclusion–matrix interface (Lin and Mura, 1973; Dunn and Taya, 1993). The fields just outside the inclusion can be expressed as

$$\Sigma^{\text{out}}_{iJ} = \Sigma^{\text{in}}_{iJ} + \widehat{E}_{iJKl}(-\widehat{E}_{pQMn}Z^{\text{T}}_{Mn}K^{-1}_{QK}n_p n_l + Z^{\text{T}}_{Kl}), \tag{47}$$

with $K_{JK} = n_i n_l \widehat{E}_{iJKl}$ and the interior fields Σ^{in}_{iJ} obtained by the approach discussed above.

Finally, we discuss some energy calculations. Consider a solid containing an inhomogeneity subjected to far-field loads $n_i\Sigma^0_{iJ}$. These loads would result in uniform fields Σ^0_{iJ} in a homogeneous solid. The total free energy of the inhomogeneity can be expressed as

$$\begin{aligned} W &= \frac{1}{2}\int_D (\Sigma^0_{iJ} + \Sigma^{\text{d}}_{iJ})(U^0_{J,i} + U^{\text{d}}_{J,i})\,\text{d}V - \int_S \Sigma^0_{iJ} n_i (U^0_J + U^{\text{d}}_J)\,\text{d}V \\ &= \frac{1}{2}\int_D \Sigma^0_{iJ} U^0_{J,i}\,\text{d}V + \frac{1}{2}\int_\Omega \Sigma^0_{iJ} Z^*_{Ji}\,\text{d}V - \int_S \Sigma^0_{iJ} n_i (U^0_J + U^{\text{d}}_J)\,\text{d}V. \end{aligned} \tag{48}$$

The interaction energy between $n_i\Sigma^0_{iJ}$ and the inhomogeneity is then

$$\Delta W = W - W^0 = \frac{1}{2}\int_\Omega \Sigma^0_{iJ} Z^*_{Ji}\,\text{d}V - \int_S \Sigma^0_{iJ} n_i U^{\text{d}}_J\,\text{d}V = -\frac{1}{2}\Sigma^0_{iJ} Z^*_{Ji} V_\Omega, \tag{49}$$

with the volume of inhomogeneity $V_\Omega = (4/3)\pi a_1 a_2 a_3$. The interaction energy is very important in the studies of ferroelectric or ferromagnetic phase transformation and domain switching.

5. Concluding remarks

We have obtained explicit expressions of the magnetoelectric Green's functions for a transversely isotropic medium exhibiting coupling between the static electric and magnetic fields utilizing the contour integral representation. The Green's function is used to analyze the magnetoelectric inclusion and inhomogeneity problems in an infinite medium, and explicit, closed form expressions are obtained for the magnetoelectric Eshelby's tensors. The magnetoelectric Eshelby's tensors can be readily used in the solution of numerous problems in the mechanics and physics of magnetoelectric solids.

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